

Superradiance Startup at Finite Temperatures of the Electromagnetic Field

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Abstract

We use quantum-electrodynamical approach to study the initial stage of Dicke superradiance from a system of two-level atoms. Applying the zeroth-order Magnus approximation, we obtain the expression for the mean number of quanta emitted in the presence of thermal fluctuations of the electromagnetic field.

Key words: Superradiance, thermal fluctuations, the Magnus expansion, terahertz.

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1 Introduction

In 1954, Dicke demonstrated that a system of N_a inverted two-level atoms interacting with an electromagnetic field can spontaneously drop to a ground state during the time proportional to N_a^{-1} [1]. This drop is accompanied by the emission of an electromagnetic-radiation pulse with the peak power P proportional to N_a^2 and is called the collective spontaneous emission, or superradiance [2].

In the classical limit, Dicke derived a formula for time dependence of the power P for two-level atom systems whose dimensions are much less than the radiation wavelength. In [3,4,5], this formula was generalized to the case of extended systems, but the suggested theories – except for the one given in [4] and discussing single-mode generation – fall short to describe the initial stage of Dicke superradiant emission in a many-atom system.

Moreover, the question remains as to the effect of thermal fluctuations of the electromagnetic field on superradiance. It is well known [6] that thermal fluctuations become essential if $kT \geq \hbar\omega$; in fact, when $kT \gg \hbar\omega$, generation starts as a stimulated emission induced by thermal quanta rather than as a spontaneous one.

In this context, our paper frames the quantum theory of superradiance startup in the presence of thermal fluctuations of the electromagnetic field. The paper is organized as follows. First, with the zeroth-order Magnus approximation [7,8] we find the expression for the mean number of photons, N , emitted by the system of two-level atoms at the onset of the generation process at finite temperatures. We shall then demonstrate that under the conditions of

a single-mode generation in the absence of thermal fluctuations, the formula for N converts to the expression derived by Bonifacio and Preparata [4].

2 Interaction of the electromagnetic field with two-level atoms

Let us consider the emission processes involving a system of two-level atoms. In the interaction representation, the Hamiltonian of the system has the form

$$\hat{H}_{int} = \sum_{\mu} \hat{a}_{\mu}^+ \hat{b}_{\mu}(t) + \hat{a}_{\mu} \hat{b}_{\mu}^+(t). \quad (1)$$

Here \hat{a}_{μ}^+ and \hat{a}_{μ} are the creation and annihilation operators of a photon with the energy $\hbar\omega_{\mu}$ at the initial time. Generally noncommuting operators $\hat{b}_{\mu}^{(+)}$ depend on time and dynamical operators characterizing the behavior of atoms. Using (1), we can easily write a formal expression for the evolution operator \hat{U} in the form suggested by Magnus [7]

$$\hat{U}_{int}(t) = \exp \left(-\frac{i}{\hbar} \int_0^t \hat{H}(x) dx - \frac{1}{2\hbar^2} \int_0^t dx \int_0^x dy [\hat{H}(x), \hat{H}(y)] + \dots \right). \quad (2)$$

Let us mention that the zeroth-order Magnus approximation was used by Becker and McIver [9], where they analyzed the photon statistics in Cherenkov oscillators and free electron lasers (the Hamiltonian used in [9] has the same form as in (1)):

$$\begin{aligned} \hat{U}_{int} &\approx e^{-\frac{i}{\hbar} \int \hat{H} dt} = e^{\sum_{\mu} \hat{a}_{\mu}^+ \hat{\alpha}_{\mu}(t) - \hat{a}_{\mu} \hat{\alpha}_{\mu}^+(t)}, \\ \hat{\alpha}_{\mu}(t) &= -\frac{i}{\hbar} \int \hat{b}_{\mu}(t) dt. \end{aligned} \quad (3)$$

The operators $\hat{b}_{\mu}^{(+)}(t)$ in [9] were assumed to be c -numbers. This only provided the description of spontaneous radiation of relativistic electrons, neglecting

the recoil effect. Naturally, the statistics of the emitted photons in this case is Poissonian, because \hat{U}_{int} at $[\hat{b}_\mu^+(t), \hat{b}_\mu(t)] = 0$ corresponds to the shift transformation responsible for a vacuum-to-Glauber state transformation [10].

Now, let us return to our problem. In the case under consideration, the rate of energy exchange between the atoms and the field is noticeably less than the characteristic oscillation frequencies ω_μ of the field. For this reason, the application of the zeroth-order Magnus approximation seems justified¹. The fact that the operators $\hat{b}_\mu^+(t)$ and $\hat{b}_\mu(t)$ are noncommuting is essential, because it is just for this noncommutativity that the radiation is enhanced, which we shall demonstrate further in this paper.

The time evolution of the annihilation operator $\hat{a}_\mu(t)$ in the zeroth-order Magnus approximation is given by

$$\begin{aligned}\hat{a}_\mu(t) &= \hat{U}_{int}^+(t)\hat{a}_\mu(0)\hat{U}_{int}(t) \\ &= \hat{a}_\mu(0) + [\hat{A}, \hat{a}_\mu(0)] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{a}_\mu(0)]] + \frac{1}{3!}[\hat{A}, [\hat{A}, [\hat{A}, \hat{a}_\mu(0)]]] + \dots, \\ \hat{A} &= -\hat{a}_\mu^+\hat{\alpha}_\mu(t) + \hat{a}_\mu\hat{\alpha}_\mu^+(t).\end{aligned}\tag{4}$$

We shall approximately consider the commutators

$$\beta_{\mu\nu}(t) = [\hat{\alpha}_\mu^+(t), \hat{\alpha}_\nu(t)] \approx Tr([\hat{\alpha}_\mu^+(t), \hat{\alpha}_\nu(t)])\tag{5}$$

to be ordinary numbers, which is true when the quantum-mechanical fluctuations of $\beta_{\mu\nu}$ are negligibly small, and assume the commutators $[\hat{\alpha}_\mu^{(+)}(t), \hat{\alpha}_\nu^{(+)}(t)]$

¹ The zeroth-order Magnus approximation describes exactly a system of two-level atoms resonantly interacting with a single-mode radiation field, since for this system the Hamiltonian in the interaction representation is time-independent [2].

to be zero.

$$[\hat{\alpha}_\mu^{(+)}(t), \hat{\alpha}_\nu^{(+)}(t)] \approx 0. \quad (6)$$

(As will be demonstrated further, the relationships (5) and (6) strictly hold at the superradiance startup.)

Then the operators $\hat{a}_\mu(t)$ readily transform to the form

$$\begin{aligned} \hat{a}_\mu(t) &= \sum_\nu \left(\hat{a}_\nu (\cosh \sqrt{\beta(t)})_{\nu\mu} + \hat{\alpha}_\nu(t) \left(\frac{\sinh \sqrt{\beta(t)}}{\sqrt{\beta(t)}} \right)_{\nu\mu} \right) \\ &= \sum_\nu \hat{a}_\nu C_{\nu\mu} + \hat{\alpha}_\nu S_{\nu\mu}. \end{aligned} \quad (7)$$

Assuming that the electric field at the initial time ($t = 0$) was at temperature T , we can find the mean number of quanta in the mode with subscript μ by formula

$$\begin{aligned} N_\mu(t) &= \langle \hat{a}_\mu^\dagger(t) \hat{a}_\mu(t) \rangle = \sum_\nu \frac{1}{\exp(\hbar\omega_\nu/kT) - 1} C_{\nu\mu}^* C_{\nu\mu} \\ &\quad + \sum_{\phi\nu} Tr(\rho_M \hat{\alpha}_\phi^+ \hat{\alpha}_\nu) S_{\phi\mu}^* S_{\nu\mu}. \end{aligned} \quad (8)$$

A remarkable feature of (8) is that it needs averaging only over the variables characterizing the state of the ensemble of elementary radiators, which can be done using the density matrix $\hat{\rho}_M$ of the atomic subsystem. Averaging over the field variables is already completed.

3 Superradiance from two-level atoms

By way of example, we shall consider in detail the startup of superradiance from a system of two-level atoms resonantly interacting with one field mode. In the interaction representation, the Hamiltonian describing this resonant interaction has the form [1,2]

$$\hat{H}_{int} = -\kappa\hat{a}\hat{R}_+ - \kappa^*\hat{a}^+\hat{R}_-, \quad (9)$$

where the atomic operators $\hat{R}_+ = \hat{R}_1 + i\hat{R}_2$ and $\hat{R}_- = \hat{R}_1 - i\hat{R}_2$ satisfy the following commutation relations [1,2]:

$$\begin{aligned} [\hat{R}_3, \hat{R}_\pm] &= \pm\hat{R}_\pm, \\ [\hat{R}_+, \hat{R}_-] &= 2\hat{R}_3. \end{aligned} \quad (10)$$

Here \hat{R}_1 , \hat{R}_2 , and \hat{R}_3 are the three projections of the pseudo-spin operator, each given by the sum of Pauli operators

$$\hat{R}_j = \sum_p \hat{\sigma}_j^{(p)}, \quad (11)$$

taken over all the atoms ($j = 1, 2, 3$).

The Hamiltonian (9) is associated with the following evolution operator

$$\begin{aligned} \hat{U}_{int} &= e^{\hat{\alpha}\hat{a}^+ - \hat{\alpha}^+\hat{a}}, \\ \hat{\alpha} &= i\kappa^*\hat{R}_-t. \end{aligned} \quad (12)$$

Using the commutation relations (10), we find the commutators needed for further calculations

$$\begin{aligned}
[\hat{\alpha}^+, \hat{\alpha}] &= 2\kappa^* \kappa \hat{R}_3 t^2, \\
[\hat{\alpha}^{(+)}, \hat{\alpha}^{(+)}] &= 0.
\end{aligned}
\tag{13}$$

The latter being equal to zero, the condition (6) is fulfilled.

Setting $\beta = 2\kappa^* \kappa T r(\hat{R}_3) t^2$ and using (8), we obtain

$$\begin{aligned}
N &= \frac{1}{\exp(\hbar\omega/kT) - 1} |C|^2 + \kappa^* \kappa t^2 |S|^2 Tr(\rho_M(\hat{R}^2 - \hat{R}_3^2 + \hat{R}_3)), \\
C &= \cosh(\sqrt{\beta}), \\
S &= \frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}.
\end{aligned}
\tag{14}$$

(15)

The operator $\hat{R}^2 = \hat{R}_+ \hat{R}_- + \hat{R}_3^2 - \hat{R}_3$ appearing in (14) commutes with \hat{H}_{int} , and so \hat{R}^2 is the integral of motion. In a system containing N_a inverted atoms, the eigenvalues of operators \hat{R}^2 and \hat{R}_3 at the superradiance startup equal $r(r+1)$ and $r = N_a/2$, respectively [1,2]. Because the system is in a state corresponding to the eigenstate of the operator \hat{R}_3 , the quantum-mechanical fluctuations of \hat{R}_3 vanish, thus supporting the validity of (5) for a system of two-level atoms.

Upon substituting the eigenvalues of the operators \hat{R}^2 and \hat{R}_3 into (14), it transforms to the form

$$N = \frac{1}{\exp(\hbar\omega/kT) - 1} \cosh^2(\sqrt{2\kappa^* \kappa r} t) + \sinh^2(\sqrt{2\kappa^* \kappa r} t).
\tag{16}$$

Let us note that at $kT \ll \hbar\omega$, the first term in (16) can be neglected, and (16) reduces to the expression derived in [4]. As follows from Fig. 1, which plots $N(t)$ against different radiation temperatures, the temperature rise is accompanied by an increase in the number of thermal quanta at $t = 0$. When

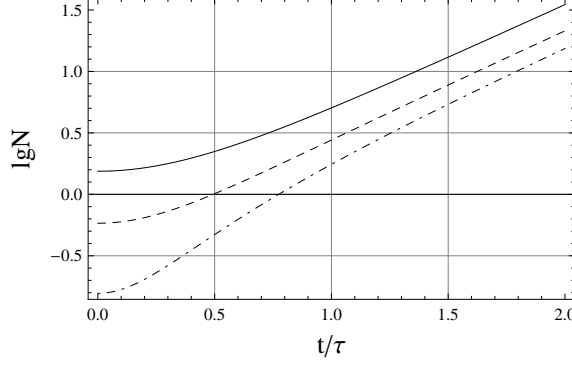


Figure 1.

$kT \gg \hbar\omega$, generation begins as the emission induced by thermal quanta rather than as a spontaneous emission. As a result, the mean number of emitted quanta at the superradiance startup is substantially increased (Fig. 1).

4 Conclusion

This paper frames the quantum theory of superradiance startup at finite temperatures of the electromagnetic field. In the zeroth-order Magnus approximation to the exponential perturbation theory, we obtained the time dependence of the mean number of emitted photons at the generation startup. In the absence of thermal fluctuations, this dependence coincides with that obtained in [4]. In the reverse case, when $kT \gg \hbar\omega$, the emission of photons becomes fundamentally different. Actually, when $kT \gg \hbar\omega$, generation begins as the emission induced by thermal quanta instead of a spontaneous emission, and this fact should be taken into account in spectroscopic studies and in development of terahertz generators [11] operating at room temperature ($kT \sim \hbar\omega_{THz}$).

The suggested theory, based on the approximate relations (3), (5), and (6), is quite general in nature, because the Hamiltonian (1) has the same structure as

the Hamiltonian describing the interaction of an arbitrary system of charged particles with a transverse electromagnetic field. For this reason, the theory framed here applies to calculating the radiation emitted by two-level atoms, as well as radiation emitted by free charged particles in Cherenkov oscillators and free electron lasers, which we shall demonstrate in subsequent works.

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Fig. 1. The number of photons as a function of time [solid curve: $\hbar\omega/kT = 0.5$, dotted curve: $\hbar\omega/kT = 1.0$, and dash-dotted curve: $\hbar\omega/kT = 2.0$, $\tau = 1/\sqrt{2\kappa^*\kappa r}$].